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Equilibrium Uniqueness in a Cournot Model with Demand Uncertainty

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Abstract

If Cournot oligopolists face uncertainty about the intercept of a linear demand function and if the realized market price must be non-negative, then expected demand becomes convex, which can create a multiplicity of equilibria. This note shows that if the distribution of the demand intercept has a monotone hazard rate and if another, rather weak, assumption is satisfied, then uniqueness of equilibrium is guaranteed.

KEYWORDS: Cournot model, non-negativity constraint, demand uncertainty, unique equilibrium

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1 Introduction

It is well known that the Cournot model may have multiple equilibria if the demand function is sufficiently convex.[1] One model environment in which the demand function, as perceived by a firm at an ex ante stage, will be convex is when the firms face uncertainty about the intercept of a linear demand function and we impose a non-negativity constraint on the realized market price. To see this, think of a situation where realized demand is linear (except for the non-negativity constraint) and the intercept can take on two possible values, low and high. In such a situation, the expected demand function will have a kink at the level of industry output where the price schedule in a low-demand state meets the horizontal axis.[2,3] In particular, the kink tends to make the expected demand schedule convex. In an accompanying paper, Lagerlöf (2006), I show that, in a simple two-state model, this convexity indeed implies the existence of two coexisting equilibria.

Given that uncertainty can have this effect, one may wonder whether there are families of distribution functions of the stochastic demand intercept for which we must have a unique equilibrium. The contribution of this note is to demonstrate that if the distribution of the demand intercept has a monotone hazard rate and if another, rather weak, assumption is satisfied, then uniqueness of equilibrium is guaranteed.

2 Model

Consider a Cournot model with $n \geq 1$ firms producing a homogeneous good. The firms are identical and indexed by $i \in \{1, 2, \ldots, n\}$. Each one of them faces a linear inverse demand function $p(X) = \max\{0, a - bX\}$, where p is price, $X \equiv \sum_{i=1}^{n} x_i$ is industry output, x_i is firm i's output, and $a > 0$ and $b > 0$ are exogenous parameters. All firms have the same constant marginal cost technology, with marginal cost denoted $c > 0$, and there is no fixed cost.

The intercept of the demand function, a, is unknown by the firms. It can take on any value in the interval $[0, a^*]$, where $a^* > 0$. (We can also have $a \in [0, \infty)$, a case which corresponds to $a^* = \infty$ in the text and the

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[1] See, for example, Vives (1999, p. 98) and the references therein.

[2] To confirm this, the reader may want to draw a figure.

[3] The fact that market price is zero for some demand realizations does not need to be interpreted literally. A richer model could assume that there are (constant unit) costs associated with selling the good. If market price falls below this cost level, the firms will prefer not to sell. In such a model, the non-negativity constraint would refer to the market price *net of selling costs*, and it could thus be binding also for a strictly positive (gross) market price. Another way of thinking about the non-negativity constraint would be that it refers to the market price *net of marginal cost*, and that there is a regulatory rule that makes a negative such net price illegal (justified by concerns for limit pricing).

formulas below.) More precisely, although the intercept a is unknown by the firms, they know that it is distributed according to the cumulative distribution function F, which is twice continuously differentiable. Its associated density function is denoted f and is strictly positive on $(0, a^*)$. Moreover, the expected value of a is assumed to exceed the constant marginal cost, $E(a) \equiv \int_0^{a^*} af(a)\,da > c$.

Each firm i is risk neutral and maximizes its expected profits. Its choice variable is its own output, $x_i \geq 0$, which it chooses simultaneously with the other firms. I will confine attention to pure strategy Nash equilibria of this game.

3 Analysis and Results

Since market price is zero for any $a \leq bX$, firm i's expected profits can be written as $E\pi_i = (P(X) - c)\,x_i$, where

$$P(X) \equiv \int_{bX}^{a^*} (a - bX)\,f(a)\,da.$$

Thus, our incomplete information model is strategically equivalent to a symmetric Cournot model with complete information and with inverse demand function $P(X)$, where $P(X)$ is continuously differentiable and with $P'(X) < 0$ in the relevant output interval. From the existing literature we know several things about such a model. In particular, the results of Amir and Lambson (2000) tell us the following.[4]

Lemma 1. *A pure strategy Nash equilibrium exists. In any such equilibrium, all firms produce the same quantity, x^*. Moreover, $x^* > 0$ and equilibrium industry output, $X^* \equiv nx^*$, satisfies*

$$\int_{bX^*}^{a^*} af(a)\,da - c = \frac{(n+1)\,bX^*}{n}\left[1 - F(bX^*)\right]. \tag{1}$$

Proof. From Theorem 2.1 of Amir and Lambson (2000) we know that, under conditions that are satisfied in our setting, our model has at least one symmetric equilibrium and no asymmetric equilibria. Moreover, by differentiating $E\pi_i$ with respect to x_i we have

$$\frac{\partial E\pi_i}{\partial x_i} = \int_{bX}^{a^*} af(a)\,da - b\,(x_i + X)\left[1 - F(bX)\right] - c. \tag{2}$$

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[4] The symmetry and existence results below actually hold for any convex cost function. In order to prove the results reported on later in this section, however, I need the constant marginal cost assumption, which is why I impose it already from the outset.

Evaluating this expression at $x_i = 0$ and $X = 0$ yields $\frac{\partial E\pi_i}{\partial x_i} \mid_{x_i=0,X=0}= E(a) - c$, which is strictly positive by assumption. Hence, all firms cannot produce zero output in a Nash equilibrium. It follows that all firms must produce the same positive quantity and that this satisfies $\partial E\pi_i/\partial x_i = 0$, which rewritten yields (1). $\square$

3.1 A Sufficient Condition for Uniqueness

The present model can have multiple equilibria. In Lagerlöf (2006), a two-state version of the model is investigated, and it is shown that, for some parameter values, it has two coexisting equilibria. The two-state distribution that was assumed in Lagerlöf (2006) could be approximated with a continuous-state two-hump distribution that satisfies all the differentiability and full-support assumptions made here, and which would therefore also give rise to multiple equilibria. It is therefore natural to ask under what circumstances the present model has a unique pure strategy Nash equilibrium. I will derive a sufficient condition for this, stated in terms of some properties of the distribution F, under the following assumption:[5]

Assumption 1. $f(0) < [E(a) - c]^{-1}$.

Rewriting firm i's first-order condition $\partial E\pi_i/\partial x_i = 0$ (using (2)) one has

$$x_i = \frac{\int_{bX}^{a^*} af(a)\,da - c}{b[1 - F(bX)]} - X \equiv R(X). \tag{3}$$

The function $R(X)$ defined in (3) is sometimes called an inclusive best-response function. It differs from a standard best-response function in that its argument is the sum of *all* firms' output, including firm i's own.[6] Although it may — because of this reason — be hard to interpret, the inclusive best-response function will prove very convenient to work with. Under our assumptions about F, R is continuous and differentiable with respect to X for all $X \in (0, a^*/b)$. Note that differentiating $R(X)$ yields

$$R'(X) = bR(X)h(bX) - 1, \tag{4}$$

where $h(X) \equiv f(X)/[1 - F(X)]$ is the hazard rate of F.

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[5] Assumption 1 is obviously satisfied for any distribution for which $f(0) = 0$, such as a log-normal. One can verify that it holds also for an exponential distribution, for a uniform distribution on $[0, a^*]$, and for the following "triangular" distribution: $f(a) = 2(a^* - a)/(a^*)^2$ for $a \in [0, a^*]$ and $f(a) = 0$ otherwise (obviously, here a^* must be finite).

[6] The inclusive best-response function was first used by Selten (1970) and then later, but independently, by Novshek (1985) to prove existence results.

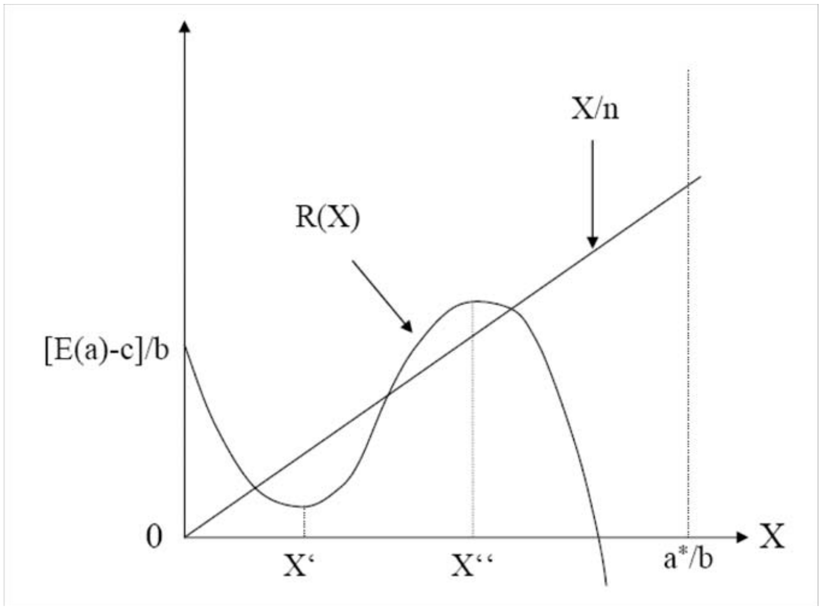

Figure 1: The shape of R(X), given that there exist more than one equilibrium.

Under Assumption 1, if there exist more than one Nash equilibrium, the graph of $R(X)$ must look something like the one that is depicted in Figure 1. More precisely, since the graph must cross the straight line X/n at least twice and it begins above it and ends below it,[7] there must exist two distinct X's, say X' and X'', with $0 < X' < X'' < a^*/b$, such that $R'(X') = R'(X'') = 0$, $R''(X') > 0$, and $R''(X'') < 0$.

What restrictions do these conditions impose on the distribution F? To see this, differentiate the expression for R' in (4):

$$R^{''}(X) = b\left[R^{'}(X)\,h(bX) + bR(X)\,h^{'}(bX)\right].$$

Hence, since $R'(X') = R'(X'') = 0$,

$$R^{''}(X') = \frac{bh^{'}(bX')}{h(bX')} \text{ and } R^{''}(X'') = \frac{bh^{'}(bX'')}{h(bX'')}.$$

This means that $h^{'}(bX') > 0$ whereas $h^{'}(bX'') < 0$.

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[7] Moreover, Assumption 1 guarantees that $R^{'}(0) < 0$.

The above analysis in conjunction with Lemma 1 yield the following proposition.

Proposition 1. *Suppose that Assumption 1 is satisfied. Moreover, suppose that F is such that its hazard rate is either (i) monotone or (ii) its slope is changing sign exactly once and it is first negative and then positive. Then there exists exactly one pure strategy Nash equilibrium.*

Most standard distribution functions have a monotone hazard rate, and it is an often-made assumption in many areas of economic theory. Of course, however, this does not necessarily mean that the condition is satisfied empirically.

What is the role of Assumption 1 in Proposition 1? As already mentioned, this assumption guarantees that the graph of $R(X)$ has a negative slope at $X = 0$, which is needed for the argument in the proof of Proposition 1 to be valid. Another way of understanding Assumption 1 is to note that it equivalently can be written as

$$\lim_{X\to 0} \frac{\partial^2 \log(P(X) - c)}{\partial X^2} < 0.$$

That is, Assumption 1 guarantees that $(P(X) - c)$ is log-concave for Xs close to zero. In his analysis of a complete information Cournot duopoly model with constant marginal costs, Amir (1996, Theorem 2.7), too, proves a uniqueness result under the assumption that $(P(X) - c)$ is log-concave, although for Xs lying in another region (namely, between the monopoly output and the output associated with marginal cost pricing). Of course, Amir's analysis is not a substitute for Proposition 1, but it suggests that Assumption 1 might not be needed for the proposition to hold.

References

Amir, Rabah (1996), "Cournot Oligopoly and the Theory of Supermodular Games," *Games and Economic Behavior* 15: 132-148.

—— and Val E. Lambson (2000), "On the Effects of Entry in Cournot Markets," *Review of Economic Studies* 67: 235-254.

Lagerlöf, Johan N.M. (2006), "Insisting on a Non-Negative Price: Oligopoly, Uncertainty, Welfare, and Multiple Equilibria," *International Journal of Industrial Organization*, forthcoming.

Novshek, William (1985), "On the Existence of Cournot Equilibrium," *Review of Economic Studies* 52: 85-98.

Selten, Reinhard (1970), *Preispolitik der Mehrproduktunternehmung in der statischen Theorie*, Berlin, Germany: Springer-Verlag.

Vives, Xavier (1999), *Oligopoly Pricing: Old Ideas and New Tools*, Cambridge, MA: MIT Press.