Supplementary Material to "A Model of Reputation in Cheap Talk, with an Application to Lobbying and Unequal Representation"

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March 12, 2006
(NOT TO BE PUBLISHED)

1 Introduction

This note contains some proofs that were omitted from Frisell and Lagerlöf (2006). The following section proves the claim made in footnote 19 in Frisell and Lagerlöf (2006). Section 3 proves the claims made about the effects of "Mandatory Registration" and "Media Scrutiny" that were made in Section 5 of Frisell and Lagerlöf (2006). Section 4 proves the claim made in footnote 12 in Frisell and Lagerlöf (2006).

2 Proof of claim in footnote 19 of Frisell and Lagerlöf (2006)

In footnote 19 of Frisell and Lagerlöf (2006) we claimed that the same qualitative results that we reported in Proposition 4 of that paper hold also with the following linear welfare function:

$$V(x_t, \theta_t) = -|x_t - \theta_t|.$$

In order to verify this it will suffice to calculate the expressions for expected first-period and second-period welfare, given this alternative welfare function, and then note that these differ from the ones in Frisell and Lagerlöf (2006) only

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by a multiplicative constant ($EV_1^{PI} = 2EW_1^{PI}$ and $EV_2^{PI} = 2EW_2^{PI}$). We do these calculations in Lemma S1 and S2 below. The proofs of the lemmas are very close analogues to the corresponding proofs in Frisell and Lagerlöf (2006).

Lemma S1. *Expected first-period welfare in a partially informative equilibrium is given by*

$$EV_1^{PI} = -\frac{1}{2} + \frac{\delta p_H}{4}.$$

Proof. There are four possible realizations of (θ_1, m_1):

$$(\theta_1, m_1) \in \{(0,0), (1,0), (0,1), (1,1)\}.$$

The event $(1,1)$ happens with probability $\frac{1}{2}[p_R + p_H + p_L\lambda_L^*]$, in which case welfare is

$$-\left|1 - \frac{1 - p_L(1-\lambda_L^*)}{1 + p_R(1-\lambda_R^*) - p_L(1-\lambda_L^*)}\right| = -\left|\frac{p_R(1-\lambda_R^*)}{1 + p_R(1-\lambda_R^*) - p_L(1-\lambda_L^*)}\right|.$$

The event $(0,1)$ happens with probability $\frac{1}{2}p_R(1-\lambda_R^*)$, in which case welfare is

$$-\left|\frac{1 - p_L(1-\lambda_L^*)}{1 + p_R(1-\lambda_R^*) - p_L(1-\lambda_L^*)}\right|.$$

By symmetry, the event $(0,0)$ happens with probability $\frac{1}{2}[p_R\lambda_R^* + p_H + p_L]$, in which case welfare is

$$-\left|\frac{p_L(1-\lambda_L^*)}{1 + p_L(1-\lambda_L^*) - p_R(1-\lambda_R^*)}\right|.$$

Finally, again by symmetry, the event $(1,0)$ happens with probability $\frac{1}{2}p_L(1-\lambda_L^*)$, in which case welfare is

$$-\left|\frac{1 - p_R(1-\lambda_R^*)}{1 + p_L(1-\lambda_L^*) - p_R(1-\lambda_R^*)}\right|.$$

Hence, expected first-period welfare can be written as

$$\begin{aligned} EV_1^{PI} &= -\frac{1}{2}[p_R + p_H + p_L\lambda_L^*]\left|\frac{p_R(1-\lambda_R^*)}{1 + p_R(1-\lambda_R^*) - p_L(1-\lambda_L^*)}\right| \\ &\quad -\frac{1}{2}p_R(1-\lambda_R^*)\left|\frac{1 - p_L(1-\lambda_L^*)}{1 + p_R(1-\lambda_R^*) - p_L(1-\lambda_L^*)}\right| \\ &\quad -\frac{1}{2}[p_R\lambda_R^* + p_H + p_L]\left|\frac{p_L(1-\lambda_L^*)}{1 + p_L(1-\lambda_L^*) - p_R(1-\lambda_R^*)}\right| \\ &\quad -\frac{1}{2}p_L(1-\lambda_L^*)\left|\frac{1 - p_R(1-\lambda_R^*)}{1 + p_L(1-\lambda_L^*) - p_R(1-\lambda_R^*)}\right|. \end{aligned}$$

Using $p_R\lambda_R^* = p_L\lambda_L^*$ and $p_H = 1 - p_R - p_L$, this simplifies to

$$\begin{aligned}
EV_1^{PI} &= -\frac{1}{2}\left[1 - p_L\left(1-\lambda_L^*\right)\right]\frac{p_R - p_L\lambda_L^*}{1+p_R-p_L} - \frac{1}{2}\left[p_R - p_L\lambda_L^*\right]\frac{1-p_L\left(1-\lambda_L^*\right)}{1+p_R-p_L} \\
&\quad -\frac{1}{2}\left[1-p_R+p_L\lambda_L^*\right]\frac{p_L\left(1-\lambda_L^*\right)}{1+p_L-p_R} - \frac{1}{2}p_L\left(1-\lambda_L^*\right)\frac{1-p_R+p_L\lambda_L^*}{1+p_L-p_R} \\
&= -\frac{\left[1-p_L\left(1-\lambda_L^*\right)\right]\left[p_R-p_L\lambda_L^*\right]}{1+p_R-p_L} - \frac{\left[1-p_R+p_L\lambda_L^*\right]p_L\left(1-\lambda_L^*\right)}{1+p_L-p_R}.
\end{aligned}$$

It is now clear from the proof of Lemma A3 in Frisell and Lagerlöf (2006) that $EV_1^{PI} = 2EW_1^{PI}$, which gives us the result. □

Lemma S2. *Expected second-period welfare in a partially informative equilibrium is given by*

$$EV_2^{PI} = -\frac{1}{2} + \frac{p_H^2}{4\left[1-\left(p_R-p_L\right)^2\right]} + \frac{p_H^2}{4\sqrt{\frac{p_H\delta}{2}\left[1-\left(p_R-p_L\right)^2\right]}}.$$

Proof. If either the L-type or R-type was drawn in the first period and the state was against it and the lobbyist chose to lie, then the lobbyist's type will be known in period 2; hence, there can be no information transmission in period 2, so welfare is $-1/2$. This happens with probability $\frac{1}{2}p_L\left(1-\lambda_L^*\right) + \frac{1}{2}p_R\left(1-\lambda_R^*\right)$.

If the above event does not happen, then there will be some information transmission in period 2. There are eight possible events. Four of these have $m_2 = 1$:

$$\left(\theta_1, \theta_2, m_2\right) \in \left\{(0,1,1), (1,1,1), (0,0,1), (1,0,1)\right\}$$

(the remaining four are identical to those above but with $m_2 = 0$). The event $(0,1,1)$ happens with probability $\frac{1}{4}p_H + \frac{1}{4}p_R\lambda_R^*$, in which case second-period welfare is

$$-\left|1 - \frac{p_H + p_R\lambda_R^*}{p_H + 2p_R\lambda_R^*}\right| = -\left|\frac{p_R\lambda_R^*}{p_H+2p_R\lambda_R^*}\right|.$$

The event $(0,0,1)$ happens with probability $\frac{1}{4}p_R\lambda_R^*$, in which case second-period welfare is

$$-\left|\frac{p_H + p_R\lambda_R^*}{p_H + 2p_R\lambda_R^*}\right|.$$

The event $(1,1,1)$ happens with probability $\frac{1}{4}p_H + \frac{1}{4}p_R$, in which case second-period welfare is

$$-\left|1 - \frac{p_H + p_R}{p_H + 2p_R}\right| = -\frac{1}{2}\left|1 - \frac{p_H}{p_H+2p_R}\right|.$$

The event $(1, 0, 1)$ happens with probability $\frac{1}{4}p_R$, in which case welfare is

$$-\left|\frac{p_H + p_R}{p_H + 2p_R}\right| = -\frac{1}{2}\left|1 + \frac{p_H}{p_H + 2p_R}\right|.$$

The four cases where $m_2 = 0$ are analogous to the ones above. Hence, the event $(1, 0, 0)$ happens with probability $\frac{1}{4}p_H + \frac{1}{4}p_L\lambda_L^*$, in which case second-period welfare is

$$-\left|\frac{p_L\lambda_L^*}{p_H + 2p_L\lambda_L^*}\right|.$$

The event $(1, 1, 0)$ happens with probability $\frac{1}{4}p_L\lambda_L^*$, in which case second-period welfare is

$$-\left|\frac{p_H + p_L\lambda_L^*}{p_H + 2p_L\lambda_L^*}\right|.$$

The event $(0, 0, 0)$ happens with probability $\frac{1}{4}p_H + \frac{1}{4}p_L$, in which case second-period welfare is

$$-\frac{1}{2}\left|1 - \frac{p_H}{p_H + 2p_L}\right|.$$

Finally, the event $(0, 1, 0)$ happens with probability $\frac{1}{4}p_L$, in which case second-period welfare is

$$-\frac{1}{2}\left|1 + \frac{p_H}{p_H + 2p_L}\right|.$$

Using the above data, we can write expected second-period welfare as

$$\begin{aligned}
EV_2^{PI} \quad = \quad & -\left[\frac{1}{2}p_L\left(1 - \lambda_L^*\right) + \frac{1}{2}p_R\left(1 - \lambda_R^*\right)\right]\frac{1}{2} - \left[\frac{1}{4}p_H + \frac{1}{4}p_R\lambda_R^*\right]\left|\frac{p_R\lambda_R^*}{p_H + 2p_R\lambda_R^*}\right| \\
& -\frac{1}{4}p_R\lambda_R^*\left|\frac{p_H + p_R\lambda_R^*}{p_H + 2p_R\lambda_R^*}\right| - \left[\frac{1}{4}p_H + \frac{1}{4}p_R\right]\frac{1}{2}\left|1 - \frac{p_H}{p_H + 2p_R}\right| - \frac{1}{4}p_R\frac{1}{2}\left|1 + \frac{p_H}{p_H + 2p_R}\right| \\
& -\left[\frac{1}{4}p_H + \frac{1}{4}p_L\lambda_L^*\right]\left|\frac{p_L\lambda_L^*}{p_H + 2p_L\lambda_L^*}\right| - \frac{1}{4}p_L\lambda_L^*\left|\frac{p_H + p_L\lambda_L^*}{p_H + 2p_L\lambda_L^*}\right| \\
& -\left[\frac{1}{4}p_H + \frac{1}{4}p_L\right]\frac{1}{2}\left|1 - \frac{p_H}{p_H + 2p_L}\right| - \frac{1}{4}p_L\frac{1}{2}\left|1 + \frac{p_H}{p_H + 2p_L}\right|.
\end{aligned}$$

The terms that do not contain λ_L^* or λ_R^* (i.e., the fourth, fifth, eighth, and ninth terms) can be rewritten as

$$\begin{aligned}
& -\left(p_H + p_R\right)\frac{1}{8}\left|1 - \frac{p_H}{p_H + 2p_R}\right| - \frac{p_R}{8}\left|1 + \frac{p_H}{p_H + 2p_R}\right| \\
& -\frac{1}{8}\left(p_H + p_L\right)\left|1 - \frac{p_H}{p_H + 2p_L}\right| - \frac{p_L}{8}\left|1 + \frac{p_H}{p_H + 2p_L}\right| \\
= \quad & -\frac{1}{8}\left(p_H + 2p_R\right) - \frac{1}{8}\left(p_H + 2p_L\right) + \frac{p_H^2}{8\left(p_H + 2p_R\right)} + \frac{p_H^2}{8\left(p_H + 2p_L\right)} \\
= \quad & -\frac{1}{4} + \frac{p_H^2}{4\left(p_H + 2p_R\right)\left(p_H + 2p_L\right)} = -\frac{1}{4} + \frac{p_H^2}{4\left[1 - \left(p_R - p_L\right)^2\right]} \qquad (1)
\end{aligned}$$

(here the last equality made use of $p_H = 1 - p_L - p_R$). The remaining terms can be rewritten as

$$-\frac{1}{4}\left[p_L\left(1-\lambda_L^*\right)+p_R\left(1-\lambda_R^*\right)\right]-\frac{1}{2}\frac{\left(p_H+p_R\lambda_R^*\right)p_R\lambda_R^*}{p_H+2p_R\lambda_R^*}-\frac{1}{2}\frac{\left(p_H+p_L\lambda_L^*\right)p_L\lambda_L^*}{p_H+2p_L\lambda_L^*}.$$

It is now clear from the proof of Lemma A4 in Frisell and Lagerlöf (2006) that $EV_2^{PI} = 2EW_2^{PI}$, which gives us the result. □

3 Proofs of claims in Section 5 of Frisell and Lagerlöf (2006)

Here we will prove some claims that were made in Section 5 of Frisell and Lagerlöf (2006). The claims are the following: (a), in terms of overall expected welfare, the institution Mandatory Registration beats the partially informative equilibrium, whenever that equilibrium exists; (b) expected first-period welfare under Media Scrutiny is given by

$$EW_1^{MS} = -\frac{1}{4} + \frac{p_H^2}{4\left(1-\Delta\right)}; \tag{2}$$

and (c), in terms of overall expected welfare, the institution Media Scrutiny beats the partially informative equilibrium, whenever that equilibrium exists.

Below we will first calculate EW_1^{MS}, thereby verifying claim (b). Then we show that expected overall welfare under Mandatory Registration exceeds expected overall welfare under Media Scrutiny. This means that in order to prove claims (a) and (c) it suffices to prove the latter claim, which we do last.

In order to calculate EW_1^{MS}, note that in period 1 there are four possibilities:

$$\left(m_1,\theta_1\right)\in\left\{(0,0),(0,1),(1,0),(1,1)\right\}.$$

The event $\left(m_1,\theta_1\right)=(1,1)$ happens with probability $\frac{1}{2}p_H+\frac{1}{2}p_R$, in which case first-period welfare is

$$-\left(1-\frac{p_H+p_R}{p_H+2p_R}\right)^2=-\left(\frac{p_R}{p_H+2p_R}\right)^2.$$

The event $\left(m_1,\theta_1\right)=(1,0)$ happens with probability $\frac{1}{2}p_R$, in which case first-period welfare is

$$-\left(\frac{p_H+p_R}{p_H+2p_R}\right)^2.$$

By symmetry, the event $\left(m_1,\theta_1\right)=(0,0)$ happens with probability $\frac{1}{2}p_H+\frac{1}{2}p_L$, in which case first-period welfare is

$$-\left(\frac{p_L}{p_H+2p_L}\right)^2.$$

And, again by symmetry, the event $(m_1, \theta_1) = (0, 1)$ happens with probability $\frac{1}{2}p_L$, in which case first-period welfare is

$$-\left(\frac{p_H + p_L}{p_H + 2p_L}\right)^2.$$

Hence, expected first-period welfare is

$$\begin{aligned}
EW_1^{MS} &= -\left[\frac{1}{2}p_H + \frac{1}{2}p_R\right]\left(\frac{p_R}{p_H + 2p_R}\right)^2 - \frac{p_R}{2}\left(\frac{p_H + p_R}{p_H + 2p_R}\right)^2 \\
&\quad -\left[\frac{1}{2}p_H + \frac{1}{2}p_L\right]\left(\frac{p_L}{p_H + 2p_L}\right)^2 - \frac{p_L}{2}\left(\frac{p_H + p_L}{p_H + 2p_L}\right)^2 \\
&= -\frac{p_R(p_H + p_R)}{2(p_H + 2p_R)} - \frac{p_L(p_H + p_L)}{2(p_H + 2p_L)} \\
&= -\frac{p_R(p_H + p_R)}{2[1 - (p_L - p_R)]} - \frac{p_L(p_H + p_L)}{2[1 + (p_L - p_R)]} \\
&= -\frac{p_R(1 - p_L)[1 + (p_L - p_R)]}{2\left[1 - (p_L - p_R)^2\right]} - \frac{p_L(1 - p_R)[1 - (p_L - p_R)]}{2\left[1 - (p_L - p_R)^2\right]} \\
&= -\frac{(p_L - p_R)[p_R(1 - p_L) - p_L(1 - p_R)] + [p_R(1 - p_L) + p_L(1 - p_R)]}{2(1 - \Delta)} \\
&= -\frac{(p_L - p_R)[p_R - p_L] + [p_R + p_L - 2p_Lp_R]}{2(1 - \Delta)} \\
&= \frac{(p_L - p_R)^2 - p_R - p_L + 2p_Lp_R}{2(1 - \Delta)} \\
&= \frac{p_L^2 + p_R^2 - p_R - p_L}{2(1 - \Delta)} = -\frac{p_R(1 - p_R) + p_L(1 - p_L)}{2(1 - \Delta)}.
\end{aligned} \tag{3}$$

In order to show that this equals the expression in (2), it is easiest to "go backwards":

$$\begin{aligned}
-\frac{1}{4} + \frac{p_H^2}{4(1 - \Delta)} &= -\frac{1}{4(1 - \Delta)}\left[\left(1 - (p_L - p_R)^2\right) - p_H^2\right] \\
&= -\frac{1}{4(1 - \Delta)}\left[1 - (p_L - p_R)^2 - (1 - p_L - p_R)^2\right] \\
&= -\frac{1}{4(1 - \Delta)}\left[-(p_L - p_R)^2 + 2(p_L + p_R) - (p_L + p_R)^2\right] \\
&= -\frac{1}{4(1 - \Delta)}\left[-2\left(p_L^2 + p_R^2\right) + 2(p_L + p_R)\right],
\end{aligned}$$

which simplifies to the expression in (3).

Let us now show that expected overall welfare under Mandatory Registration exceeds expected overall welfare under Media Scrutiny. Using the relevant expressions from Section 5 of Frisell and Lagerlöf (2006) we have that this is true if and only if

$$-\frac{(1 - p_H)(1 + \delta)}{4} > -\frac{1}{4} + \frac{p_H^2}{4(1 - \Delta)} - \frac{\delta(1 - p_H)}{4} \Leftrightarrow 1 - \Delta > p_H.$$

Using $\Delta \equiv (p_L - p_R)^2$ and $1 - p_H \equiv p_L + p_R$, this simplifies to

$$p_L + p_R > (p_L - p_R)^2 \Leftrightarrow p_L(1 - p_L) + p_R(1 - p_R) > -2p_L p_R,$$

which always holds since $p_L, p_R \in (0, 1)$.

Finally, let us show that, in terms of overall expected welfare, the institution Media Scrutiny beats the partially informative equilibrium, whenever that equilibrium exists. By using the expressions for expected welfare that are stated in Sections 4 and 5 of Frisell and Lagerlöf (2006), we have that this is true if

$$-\frac{1}{4}+\frac{p_H^2}{4(1-\Delta)}-\frac{\delta(1-p_H)}{4} > -\frac{1}{4}+\frac{\delta p_H}{8}+\delta\left[-\frac{1}{4}+\frac{p_H^2}{8(1-\Delta)}+\frac{p_H^2}{8\sqrt{\frac{p_H\delta(1-\Delta)}{2}}}\right]$$

or, simplifying,

$$\frac{2p_H}{1-\Delta}+2\delta > \delta+\delta\left[\frac{p_H}{1-\Delta}+\frac{p_H}{\sqrt{\frac{p_H\delta(1-\Delta)}{2}}}\right].$$

Re-arranging, we have

$$\frac{p_H(2-\delta)}{1-\Delta} > \delta\left[\frac{p_H}{\sqrt{\frac{p_H\delta(1-\Delta)}{2}}}-1\right] = \delta\sqrt{p_H}\left[\frac{\sqrt{p_H}-\sqrt{\frac{\delta(1-\Delta)}{2}}}{\sqrt{\frac{p_H\delta(1-\Delta)}{2}}}\right].$$

But the definition of λ_L^* (see (4) in Frisell and Lagerlöf (2006)) and the requirement that $\lambda_L^* > 0$ imply that the numerator of the last ratio is strictly negative. Hence, since the left-hand side is positive for all $\delta \in (0, 1]$, the equality must hold.

4 Proof of claim in footnote 12 of Frisell and Lagerlöf (2006)

In footnote 12 of Frisell and Lagerlöf (2006) we claimed that, under the assumption that $\delta \leq 1$, $\lambda_L^* < 1$ and $\lambda_H^* < 1$. Here we will prove that $\lambda_L^* < 1$ (the proof that $\lambda_H^* < 1$ is analogous).

$$
\begin{aligned}
\lambda_L^* &= \frac{1}{2p_L}\left[\sqrt{\frac{\delta\left[1-(p_R-p_L)^2\right]}{2}p_H}-p_H\right] < 1 \\
&\Leftrightarrow \sqrt{\frac{\delta\left[1+(p_R-p_L)\right]\left[1-(p_R-p_L)\right]}{2}p_H} < 2p_L+p_H = 1-p_R+p_L \\
&\Leftrightarrow \sqrt{\frac{\delta\left[1+(p_R-p_L)\right]}{2}p_H} < \sqrt{1-p_R+p_L} \\
&\Leftrightarrow \frac{\delta\left[1+(p_R-p_L)\right]}{2}\left(1-p_R-p_L\right) < 1-p_R+p_L.
\end{aligned}
$$

The left-hand side is decreasing in p_L and the right-hand side is increasing in p_L, so set $p_L = 0$ to get the most restrictive case. The inequality then reads

$$\frac{\delta\left(1+p_R\right)}{2}\left(1-p_R\right) < 1-p_R \Leftrightarrow \delta\left(1+p_R\right) < 2,$$

which always holds.

Reference

Frisell, Lars and Johan N.M. Lagerlöf (2006). "A Model of Reputation in Cheap Talk, with an Application to Lobbying and Unequal Representation," mimeo, Sveriges Riksbank, Stockholm and Royal Holloway, London.